Essays in the history of the theory of structures
In honour of Jacques Heyman

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The 1826 publication of Navier’s *Leçons*, the design of early suspension bridges, and the design of iron box girder railroad bridges exemplify the transforming changes in the design of structures that began in the first half of the nineteenth century. Navier’s text demonstrates that linear elastic structural analyses provide essential insights on structural behavior and may serve as bases for an allowable stress design approach. For example, Navier gives the general solution for the forces in a king-post truss and observes that a parallel-chord truss may be viewed as a beam with an effective moment of inertia proportional to the chord areas times the square of the distance between the chords. This basic analogy formed the basis for the design of truss chords in the US in the 1820’s. Navier also provides an analytical solution to what may be viewed as the “elementary hyperstatic system”, the three bar truss shown as figure 112 in plate IV of his text. Navier writes the joint equilibrium equations, the linear elastic force-deformation equations, and the small-displacement geometric compatibility equations between the element axial deformations and the joint displacements. Navier then uses a “stiffness formulation” to solve for the joint displacements and the member forces. Navier in all likelihood understood the increase in computational effort required for analyses of trusses with many joints. Implicit in Navier’s analysis are deterministic (zero) displacement boundary conditions and the assumption that the members fit perfectly. That is, Navier did not consider any self-equilibrated forces in the hyperstatic truss prior to the application of the load.

The early suspension bridges built in Britain, France and the United States posed challenging new design problems for engineers. Small-span suspension
bridges have live-load-to-dead-load ratios significantly larger than, say, those of stone arches. Moreover, hanging cables or chains have small vertical stiffnesses for some live load conditions. Therefore design live loads models were debated intensely. Navier argued for a uniform design live load of three 65 kg persons per square meter, or 200 kg/m², while Marc Seguin reasoned that in cases where the population was small and the traffic light, “it is quite possible to lower the requirement, as the probability of such a load is so small as to be considered nonexistent” (Peters 1987). Seguin’s concept of reducing the live load based on the probability of occurrence continues today in both building and bridge design. Designers agreed that adequate stiffness was a critical design criterion and numerous conceptual designs of stiffening systems were proposed and tried (Gasparini et al. 1999). Navier’s advanced understanding of the effect of the cable axial force on its vertical stiffness formed his judgment that stiffness is most easily achieved by using small sag-to-span ratios. In addition to issues of appropriate loads and stiffness, suspension bridge design also initiated discussions on reliability, specifically on the relative short and long term reliabilities of chains versus wire cables. Navier did not favor wire cables because he believed that they were susceptible to corrosion over time. On the other hand, designers understood that the strength of a chain depends on its weakest link.

Robert Stephenson’s concept of riveted iron box girder railroad bridges raised other design issues. For one, the stability of plates in compression had to be assured. Most importantly, there arose a need for a rational basis for design of riveted connections. A riveted (or bolted) connection is the bete noir of a linear elastic analysis/design approach. Stress states depend on unknown boundary conditions and on friction from the prestressing of the rivets and plates. Fairbairn (1849) and Hodgkinson advocated a strength design approach; that is, proportioning a connection such that its strength is safely greater than working loads (or, say, equal to factored loads). Stephenson and Clark (1850) advocated designs that gave satisfactory performance, say no slip, at service loads. Since it was not possible to predict strength analytically, Fairbairn and Hodgkinson performed extensive tests, which revealed considerable statistical scatter in strengths. Fairbairn based his “100-75-56” rule on observed mean strengths.

These and other nineteenth century design experiences with new materials and structural forms made engineers more aware of uncertainties in loads and strengths and of the need to provide both sufficient strength and satisfactory performance at service conditions. In the absence of analytical methods and testing resources for predicting strength, design methods based on linear elastic analyses became dominant. These, of course, involve defining very conservative “working” loads, performing linear elastic analyses, and “allowing” stresses that are
fractions of very conservative ("minimum") strengths. Such methods have produced safe designs and remain predominant. The computational effort of performing analyses was decreased by manual iterative methods for solving simultaneous equations and then revolutionized by computer-based finite element formulations. However, Heyman (1995) notes that computed stresses in some hyperstatic systems are "illusory", because of uncertain force and displacement boundary conditions. Further, Heyman observes that such uncertainties generally do not affect the strength of systems and therefore he advocates the use of mechanism limit state analyses for certain systems.

Within the past forty or so years, considerable research has been performed on nondeterministic analysis and design methods. Such methods provide a rational basis for defining design criteria for exceptional projects such as "lifelines", power plants, etc. Because some of the uncertainty is modeled, nondeterministic methods have the potential to improve decision-making and design. Estimating reliability and achieving a design with a prescribed reliability are rational design objectives. The objective here is not to provide a chronological account of individual developments in nondeterministic methods. Rather, it is to present basic nondeterministic methods for static structural analysis, discuss insights that they may provide, contrast them with deterministic methods, and discuss possible associated design methods. Navier's elementary hyperstatic system, with specific direction angles as shown in figure 1, is used to illustrate ideas and concepts. Deterministic linear elastic analyses are reviewed first. Then three nondeterministic linear elastic analysis approaches are discussed. Next, deterministic and nondeterministic mechanism limit state analyses are compared, considering both ductile and brittle elements.

![Diagram](image-url)

**Figure 1**
Navier's elementary hyperstatic system with specific direction angles
Linear elastic analyses. Deterministic systems and loads

The most common method for linear elastic static analysis involves writing nodal equilibrium equations in the matrix stiffness form:

$$KU = P$$  \hspace{1cm} (1)

In which $K$ is the system stiffness matrix, $U$ is the vector of nodal displacements, and $P$ is the vector of effective nodal loads. If the system is considered to be deterministic, $K$ is a matrix of real numbers. $K$ is typically assembled by adding stiffness contributions from individual elements; this process reveals that the stiffness of elements or subsystems in parallel are additive. The effective nodal load vector, $P$, may be expressed as:

$$P = MQ$$  \hspace{1cm} (2)

in which $Q$ is a vector of magnitudes for any specific load system and $M$ is a matrix whose rows are contributions to effective nodal loads from unit values of the forces in $Q$. Figure 2 shows example $Q$ vectors for a truss and a plane frame.

Figure 2
Load magnitude vectors, $Q$, for two specific load systems.
After imposing displacement boundary conditions, the nodal displacement vector is, symbolically:

$$U = (K^{-1}M)Q$$

(3)

In turn, any vector of element stress resultants, $N$, may be expressed as a linear function of $U$,

$$N = BU = (BK^{-1}M)Q = A_q Q$$

(4)

Eqs. (3) and (4) simply indicate that for a linear elastic system, any load effect vector may always be expressed as a linear function of a load magnitude vector, $Q$. The matrices $K^{-1}M$ and $BK^{-1}M = A_q$ may have important physical meaning for specific load systems. For example, for the $Q$ vector shown in figure 2a, $K^{-1}M$ is a matrix whose rows are influence line values for nodal displacements and $BK^{-1}M = A_q$ is a matrix whose rows are influence line values for element axial forces. For the $Q$ vector of figure 2b, the rows of $BK^{-1}M = A_q$ show the contributions of each bay loading to the stress resultants in $N$. Therefore the signs of the terms in any row of $A_q$ indicate which bays must be loaded to obtain extreme values of the corresponding stress resultant. For the Navier system shown in figure 1, the matrix $M$ is simply a unit diagonal matrix and a simple stiffness analysis gives:

$$U = \begin{bmatrix} u_i \\ v_j \end{bmatrix} = (K^{-1}M)Q = \frac{1}{1.8672K} \begin{bmatrix} 1.0079 & -0.3750 \\ -0.3750 & 1.9921 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

(5a)

$$N = \begin{bmatrix} N_i \\ N_j \end{bmatrix} = BU = K \begin{bmatrix} 12/13 & 5/13 \\ 1/\sqrt{2} & 1/\sqrt{2} \\ -4/5 & 3/5 \end{bmatrix} \begin{bmatrix} u_i \\ v_j \end{bmatrix}$$

(5b)

$$N = \begin{bmatrix} 0.4211 & 0.2249 \\ 0.2397 & 0.6124 \\ -0.5524 & 0.8008 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = A_q Q$$

(5c)

In which $K = EA/L$ is the axial stiffness of all three elements. If a structure is statically determinate, $A_q$ is independent of material and element properties. Although the truss is hyperstatic, the element axial forces are independent of the element axial stiffnesses because all the axial stiffnesses are the same. Of course if the element axial stiffnesses were not equal, any element axial force (in a
hyperstatic system) is proportional to that element's stiffness contribution. The linear elastic analysis represented by Eqs. (5) assumes zero lack-of-fit and zero prescribed initial support displacements. But both lack-of-fit and non-zero prescribed support displacements may be considered as actions or loads. Let \( \mathbf{e}_0 = [e_{10}, e_{20}, e_{30}]^T \) be a vector of initial element axial deformations and \( \mathbf{U}_0 = [u_{10}, v_{10}, u_{20}, v_{20}, u_{30}, v_{30}]^T \) be a vector of non-zero prescribed support displacements. Then a linear elastic analysis gives:

\[
\begin{bmatrix}
N_1 \\
N_2 \\
N_3
\end{bmatrix} = K \begin{bmatrix}
0.5248 & 0.4568 & -0.2017 \\
0.4568 & -0.3975 & 0.1757 \\
-0.2019 & 0.1757 & -0.0776
\end{bmatrix} \mathbf{e}_0 + K \begin{bmatrix}
-0.4844 & -0.2018 & 0.3230 & 0.3230 & 0.1615 & -0.1201 \\
0.4217 & 0.1757 & -0.2811 & -0.2811 & -0.1406 & 0.1054 \\
-0.1863 & -0.0776 & 0.1242 & 0.1242 & 0.0616 & 0.0462
\end{bmatrix} \mathbf{U}_0 \quad (6a)
\]

\[\mathbf{N} = \mathbf{A}_e \mathbf{e}_0 + \mathbf{A}_a \mathbf{U}_0 \quad (6b)\]

The matrices \( \mathbf{A}_e \) and \( \mathbf{A}_a \) must be identically zero for statically determinate systems. For the hyperstatic Navier truss, the axial forces from \( \mathbf{e}_0 \) and \( \mathbf{U}_0 \) are proportional to the element axial stiffnesses, \( K_e = K \). For a linear elastic system, the effects of the three actions may be superposed:

\[\mathbf{N} = \mathbf{A}_e \mathbf{e}_0 + \mathbf{A}_a \mathbf{U}_0 + \mathbf{A}_q \mathbf{Q} \quad (7)\]

Effects of changes in temperature may also be added to Eq. (7). It should be noted that, thus far, no models for the action vectors, \( \mathbf{Q} \), \( \mathbf{e}_0 \), and \( \mathbf{U}_0 \) have been introduced. That is, Eqs. (1) to (7) are valid for deterministic system models and for both deterministic and nondeterministic load or action models. For deterministic linear elastic analyses, the actions are modeled as vectors of real numbers. Then any response vector will also be a vector of real numbers. For example, if \( \mathbf{Q} = [10 \ 50]^T \) and \( K = 30,000 \), then \( \mathbf{U} = [0.000155 \ 0.00171]^T \) and \( \mathbf{N} = [15.46 \ 33.01 \ 34.53]^T \).

Observations. Deterministic linear elastic analyses give "point estimates" of responses of models of structures to models of loads or actions. Whether the estimates truly predict actual structural response depends on the quality or "goodness" of the system and load models. A designer may always try to refine or improve the structural system model. For example, if there are concerns about the rotational stiffness of the nodes or, say, the flexibility of the supports, a frame model may be defined as shown in figure 3.
Deterministic linear elastic analyses with refined or improved models still give only point estimates of responses. Deterministic linear elastic analyses do provide checks of system and element serviceability criteria and indicate rational ways of increasing system stiffness or decreasing system responses. Deterministic linear elastic analyses do not provide information on strength, except, if equilibrium is satisfied with stresses at or below yield and no mechanism exists, actual strengths of ductile systems must be greater than the working loads used for linear elastic analysis. Deterministic linear elastic analyses do not provide information on the possible range of responses. If the linear elastic material model is appropriate, linear elastic analyses may be applied to most systems using available finite element programs. However, there are systems such as bolted connections, welded connections, cracked masonry, or cracked concrete systems where uncertainties in the details or boundary conditions make linear elastic analyses “nonpredictive.”

**Linear elastic analyses. Deterministic systems, nondeterministic loads**

As noted, Eqs. (1) to (7) are valid for linear elastic deterministic systems and for both deterministic and nondeterministic load models. Three principal nondeterministic models for a scalar load, $Q$, are shown in figure 4. Figure 4b depicts a random variable model; figure 4c depicts an interval number (Alefeld 1983) model; and figure 4d depicts a fuzzy number (Zadeh 1965; Kaufman and Gupta
1985) model. To define vector random variables, multidimensional probability density functions are generally required. Multidimensional probability density functions are difficult to define and use; however, there are two important exceptions. If a set of random variables are independent, then a multidimensional probability density function is simply a product of one-dimensional density functions. If the random variables are normal or Gaussian, then the multidimensional probability density function is fully defined by two statistical moments, the mean and covariance matrices. Even if the random variables are not independent or Gaussian, it is common to perform analyses with only these two moments, using "second moment" algebra. The mean and covariance matrices of a random vector, $Q$, are defined by:

$$
\text{Mean matrix } = E[Q] = \mu_Q
$$

$$
\text{Covariance matrix } = E[(Q - \mu_Q)(Q - \mu_Q)^T] = \Sigma_Q
$$

In which $E[\cdot]$ denotes the expectation operator.

If only the above two statistical moments are known, $Q$ is referred to as a "second moment" vector, and is denoted as: $Q \sim (\mu_Q; \Sigma_Q)$. A generic term of the covariance matrix is: $\text{Cov} [Q_iQ_j] = E [(Q_i - \mu_{Q_i})(Q_j - \mu_{Q_j})]$. The $\text{Cov} [Q_iQ_j]$ may

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**Figure 4**
The deterministic and three nondeterministic models for a scalar load, $Q_i$. 

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a) $Q_i$ is a real number

b) $Q_i$ is a random variable

c) $Q_i$ is an interval number

d) $Q_i$ is a fuzzy number
be expressed in terms of a correlation coefficient, $\rho_{QQ}$, between $Q_i$ and $Q_j$ as follows:

$$\text{Cov}[Q_i, Q_j] = \rho_{QQ} \sqrt{\text{Var}[Q_i]} \sqrt{\text{Var}[Q_j]}$$  \hspace{1cm} (10)

$\rho_{QQ}$ is a measure of the linear correlation between two random variables, a concept not contained in deterministic analyses. It follows from Eq. 5c that a linear elastic analysis of a deterministic system is simply a linear transformation between a load vector, $Q$, and a response vector, $N$. From properties of the expectation operator, the mean and covariance matrices of the response vector are:

$$E[N] = \mu_N = A_q \mu_q$$  \hspace{1cm} (11)

$$E[(N - \mu_N)(N - \mu_N)^\top] = \Sigma_{NN} = A_q \Sigma_{QQ} A_q^\top$$  \hspace{1cm} (12)

Therefore the mean and covariance matrices of any response vector of a linear elastic deterministic system are found using simple matrix multiplication. For example, assume the load vector for the Navier system of figure 1 is defined by:

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = Q - \begin{bmatrix} 10 \\ 50 \end{bmatrix}; \quad \begin{bmatrix} 4 & 15 \\ 15 & 225 \end{bmatrix}$$  \hspace{1cm} (13)

Then, using Eqs. 11 and 12, and assuming $K = 30,000$, $U$ and $N$ are second moment vectors:

$$\begin{bmatrix} u_i \\ v_j \end{bmatrix} = U - \begin{bmatrix} 0.000155 \\ 0.00171 \end{bmatrix}; \quad \begin{bmatrix} 0.7767 \times 10^{-4} & -4.378 \times 10^{-8} \\ -4.378 \times 10^{-8} & 27.76 \times 10^{-8} \end{bmatrix}$$  \hspace{1cm} (14)

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = N - \begin{bmatrix} 15.46 \\ 33.01 \end{bmatrix}; \quad \begin{bmatrix} 14.94 & 36.08 & 42.81 \\ 36.08 & 89.00 & 107.67 \\ 42.81 & 107.67 & 132.31 \end{bmatrix}$$  \hspace{1cm} (15)

The coefficient of variation of a random variable is defined as $\sqrt{\text{Var}[N_i]} / E[N_i]$. The coefficients of variation of $N = [N_1, N_2, N_3]^\top$ are 0.25, 0.285, and 0.333 respectively. In general, each response has a different uncertainty. The moments of $N$ given in Eq. 15 were computed for the assumed correlation between $Q_1$ and $Q_2$ contained in the covariance matrix in Eq. 13. $\Sigma_{NN}$ of Eq. 15 indicates that the
three axial forces are highly correlated. The effects of changing the correlation between \( Q_1 \) and \( Q_2 \) on the moments of \( N \) are easily quantified using Eq. 12. Even if the loads are uncorrelated, the axial forces are correlated because they are all functions of the same loads. In other words, a diagonal load covariance matrix, \( \Sigma_{qq} \), still results in a full response covariance matrix, \( \Sigma_{rr} \), because of the matrix product in Eq. 12.

The three actions, \( Q_e \), \( e \), and \( U_0 \) may be combined in a total action vector, \( Q_T \), defined by:

\[
Q_T = \begin{bmatrix} Q_0 \\ Q_e \\ U_0 \end{bmatrix} = \begin{bmatrix} \Sigma_{qq} & \Sigma_{q_e} & \Sigma_{q_u} \\ \Sigma_{e_q} & \Sigma_{ee} & \Sigma_{eu} \\ \Sigma_{u_q} & \Sigma_{eu} & \Sigma_{uu} \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_e \\ U_0 \end{bmatrix}
\]

(16)

The moments of \( N = [A_x A_y A_z] \) \( Q_T = \Lambda_T Q_T \) may be computed using Eqs. 11 and 12. This computation is simplified if it is assumed that \( \Sigma_{qq} = 0 \), \( \Sigma_{q_e} = 0 \), and \( \Sigma_{eu} = 0 \); that is, if all pairs of actions from two different action vectors are assumed to be uncorrelated.

In lieu of modeling \( Q \) as a second moment vector, it may be modeled as a vector of interval numbers, with each component defined by upper and lower bounds. For example, for the loads on the Navier truss:

\[
Q = \begin{bmatrix} [Q_0^1; Q_0^2] \\ [Q_e^1; Q_e^2] \\ [Q_u^1; Q_u^2] \end{bmatrix}
\]

(17)

In which \( Q_0^1 \) and \( Q_0^2 \) are the upper and lower bounds of \( Q \). Any response vector of a linear elastic system is also a vector of interval numbers. For the Navier truss:

\[
N = \begin{bmatrix} \sum_{j=1}^{2} \min(a_x Q_{0j}, a_y Q_{0j}, a_z Q_{0j}) \\ \sum_{j=1}^{2} \max(a_x Q_{0j}, a_y Q_{0j}, a_z Q_{0j}) \\ \sum_{j=1}^{2} \min(a_x Q_{ej}, a_y Q_{ej}, a_z Q_{ej}) \\ \sum_{j=1}^{2} \max(a_x Q_{ej}, a_y Q_{ej}, a_z Q_{ej}) \\ \sum_{j=1}^{2} \min(a_x Q_{uj}, a_y Q_{uj}, a_z Q_{uj}) \\ \sum_{j=1}^{2} \max(a_x Q_{uj}, a_y Q_{uj}, a_z Q_{uj}) \end{bmatrix}
\]

(18)

In which \( a_y \) are the coefficients of the \( A_y \) matrix in Eq. 5c. Assuming the bounds of the interval numbers in \( Q \) are (these assumed bounds correspond to "mean ± two standard deviations" as given in Eq. 13):
\[ Q = \begin{bmatrix}
6 & 14 \\
20 & 80 \\
\end{bmatrix} \]  \hspace{1cm} (19)

Then, the interval vector, \( N \), is:
\[ N = \begin{bmatrix}
7.026 & 23.89 \\
13.686 & 52.344 \\
8.287 & 60.766 \\
\end{bmatrix} \]  \hspace{1cm} (20)

\( Q \) may also be modeled as a vector of fuzzy numbers. A fuzzy number may be viewed as a set of interval numbers, each with different bounds corresponding to a different “level of presumption”, \( \alpha \). Figure 5d shows a “triangular” fuzzy number. The interval bounds associated with a specific level of presumption, \( \alpha \), are denoted as \( \alpha N_i \) and \( \alpha Q_i \). If the load vector is modeled as a fuzzy vector, then any response vector will also be a fuzzy vector. For example, an axial force response of the Navier system at a specific level of presumption, \( \alpha \), denoted by \( \alpha N \), is an interval number with bounds:
\[ \left[ \alpha N^l; \alpha N^u \right] = \left[ \sum_{j=1}^{2} \min(a_{ij}, a_{ij}Q^l), \sum_{j=1}^{2} \max(a_{ij}, a_{ij}Q^u) \right] \]  \hspace{1cm} (21)

If it is assumed that \( Q \) and \( Q \) are fuzzy numbers defined by figure 5a and 5b, then the element force \( N \) is also a fuzzy number that may be computed by repeated use of Eq. 21 for various values of \( \alpha \). The result is shown in figure 5c.

**Observations.** A load vector, \( Q \), may be modeled as a second moment random vector defined by its mean and covariance matrices. The covariance matrix of \( Q \) should be assembled from load data, which should include information on the linear correlation between load components. Mean and covariance matrices of

![Figure 5](https://via.placeholder.com/150)

Fuzzy number models for \( Q_1 \) and \( Q_2 \); fuzzy number response, \( N_3 \).
responses computed using Eqs. 12 and 13 may be considered to be conditional on the deterministic system model that is being used. Therefore the computed statistical moments of responses do not fully capture all of the possible response uncertainty. Nonetheless, response moments reflect the effects of correlation between loads and provide estimates of the coefficient of variation of all responses. The analyses do not predict extreme values of responses and their associated probabilities of nonexceedance, unless it is assumed that all random variables are normal.

An interval vector load model does not contain the concept of linear correlation between components. However, response bounds may be computed directly. The computation does not involve multiplication of interval numbers, only addition/subtraction and multiplication by constants are required. The arithmetic of adding interval bounds directly provides maximum and minimum responses from “pattern loads”.

As for deterministic linear elastic analyses, nondeterministic linear elastic analyses do not predict strength and, for systems with unknown details or boundary conditions, they may be “non-predictive”.

Linear elastic analyses. Nondeterministic systems and loads

Although a structural model may have been defined well and may be predictive of the behavior of an actual structure, the computed responses depend on parameters that may not be deterministic. For example, for basic linear elastic frame models, the modulus of elasticity or the element properties such as areas and moments of inertia may be uncertain. The nodal coordinates may also have some uncertainty, although usually it is not significant. Even if a model has been “refined”, say by modeling connection behavior and support flexibility as shown in figure 3, the additional model parameters may also be uncertain. As for loads, system parameters may be modeled nondeterministically using random variables, interval numbers, or fuzzy numbers. Formulations that use random variables and random fields are usually called “stochastic finite elements” (Vanmarcke and Grigoriu 1983; Sparos and Ghanem 1989; Deodatis and Shinozuka 1991); formulations that use interval or fuzzy number models are called “interval finite elements” (Köylüoğlu and Elishakoff 1998) or “fuzzy finite elements” (Muñana and Mullen 1999).

In stochastic finite element formulations the usual assumption made is that the nodal coordinates are deterministic and that the principal sources of uncertainty are the “EA” and “EI” terms in the stiffnesses. The additional considerations that arise from treating the EA and EI terms as random variables may be illustrated by considering the Navier truss. The system equilibrium equations in stiffness form are:
\[
\begin{bmatrix}
\frac{144}{169} K_1 + \frac{1}{2} K_2 + \frac{16}{25} K_3 \\
\frac{60}{169} K_1 + \frac{1}{2} K_2 - \frac{12}{25} K_3 \\
\frac{60}{169} K_1 + \frac{1}{2} K_2 - \frac{12}{25} K_3
\end{bmatrix}
\begin{bmatrix}
u_i \\
u_j \\
u_l
\end{bmatrix}
= \begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
u_i \\
u_j
\end{bmatrix} = MQ
\] (22)

In which \( K_i = E_i A_i / L_i \) are the element axial stiffnesses. Terms of both the stiffness matrix and the load vector are now random variables. The element axial stiffnesses are now treated as random variables, which may be modeled as correlated second moment variables:

\[
\begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix}
\sim \begin{bmatrix}
\mu_{K_1} \\
\mu_{K_2} \\
\mu_{K_3}
\end{bmatrix}
\begin{bmatrix}
\text{var}[K_1] & \text{cov}[K_1, K_2] & \text{cov}[K_1, K_3] \\
\text{cov}[K_1, K_2] & \text{var}[K_2] & \text{cov}[K_2, K_3] \\
\text{cov}[K_1, K_3] & \text{cov}[K_2, K_3] & \text{var}[K_3]
\end{bmatrix}
\] (23)

The three system stiffnesses, written in vector form as \([K_{11}, K_{12}, K_{22}]^T\), are simply linear combinations or sums of the stiffness contributions of the elements:

\[
\begin{bmatrix}
K_{11} \\
K_{12} \\
K_{22}
\end{bmatrix}
= \begin{bmatrix}
\frac{144}{169} & \frac{1}{2} & \frac{16}{25} \\
\frac{60}{169} & \frac{1}{2} & -\frac{12}{25} \\
\frac{60}{169} & \frac{1}{2} & -\frac{12}{25}
\end{bmatrix}
\begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix}
\] (24)

Therefore the vector of stiffnesses that constitute the system stiffness matrix may be directly defined as a second moment vector using Eqs. 12 and 13. To compute moments of the displacement responses, the stiffness matrix may be inverted:

\[
\begin{bmatrix}
u_i \\
u_j
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{K_{11} - K_{12}^2/K_{22}} & \frac{1}{K_{12}^2 - K_{11} K_{22}/K_{12}} \\
\frac{1}{K_{12}^2 - K_{11} K_{22}/K_{12}} & \frac{1}{K_{22} - K_{12}^2/K_{11}}
\end{bmatrix}
\begin{bmatrix}
u_i \\
u_j
\end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\
f_{21} & f_{22} \end{bmatrix}
\begin{bmatrix}
u_i \\
u_j
\end{bmatrix} = K^{-1}MQ
\] (25)
The system flexibilities, written in vector form as \( f'_{11}f'_{12}f'_{22} \), are also a second-moment vector. Each flexibility is an inverse of a “reduced stiffness”. The matrix inverse is a nonlinear operation; that is, the system flexibilities are not obtained by a linear transformation of the system stiffnesses. Therefore moments of the vector of system flexibilities must be estimated, typically by Taylor series expansions of the flexibilities about the mean values of the stiffnesses or by Monte Carlo simulation. Eq. 25 shows that the flexibilities are multiplicatively coupled with the load random variables. It is generally reasonable to assume that system stiffnesses or flexibilities are uncorrelated with the applied loads, \( Q \). Therefore statistical moments of the nodal displacement vector may be computed using second moment algebra. The element forces may be expressed as:

\[
N = \begin{bmatrix}
K_1 & 0 & 0 \\
0 & K_2 & 0 \\
0 & 0 & K_3
\end{bmatrix} \begin{bmatrix}
12/13 & 5/13 \\
1/\sqrt{2} & 1/\sqrt{2} \\
-4/5 & 3/5
\end{bmatrix} \begin{bmatrix}
f_{11} \\
f_{12} \\
f_{21} \\
f_{22}
\end{bmatrix} MQ
\] (26)

Therefore, to compute statistical moments of the element force vector, \( N \), the covariances between the element axial stiffnesses and the system flexibilities are needed. In lieu of these steps, Monte Carlo simulation may be used to estimate moments of the nodal displacements and element force responses. For the Navier truss, the vector of basic random variables is \( [K_1, K_2, K_3, Q_1, Q_2]^T \). It may be defined as a second moment vector. For example, consider the following specific values:

\[
\begin{bmatrix}
30,000 \\
30,000 \\
30,000 \\
10 \\
50
\end{bmatrix} \begin{bmatrix}
2.5 \times 10^6 & 1.5 \times 10^6 & 1.5 \times 10^6 & 0 & 0 \\
1.5 \times 10^6 & 2.5 \times 10^6 & 1.5 \times 10^6 & 0 & 0 \\
1.5 \times 10^6 & 1.5 \times 10^6 & 2.5 \times 10^6 & 0 & 0 \\
0 & 0 & 0 & 4 & 15 \\
0 & 0 & 0 & 15 & 225
\end{bmatrix}
\] (27)

The above assumed moments of \( [K_1, K_2, K_3, Q_1, Q_2]^T \) imply that the loads are uncorrelated with the stiffnesses, that all stiffnesses have the same mean, variance, and coefficient of variation, 0.053, and that the stiffnesses are positively correlated with \( \rho_{KQ} = 0.6 \). Monte Carlo simulation then involves assuming a density function for the random variables, generating realizations of correlated random variables whose moments match the prescribed mean and covariance matrices, performing linear elastic analyses, obtaining realizations of the responses, \( U \) and \( N \), and computing statistical estimates of \( E[U] \), \( E[N] \), \( \Sigma_{UU} \) and \( \Sigma_{NN} \).
Figure 6
Nodal displacements - nondeterministic system, deterministic loads

Figure 7
Element axial forces - nondeterministic system, deterministic loads
Simulations. Results of two Monte Carlo simulations are presented to illustrate effects of random system parameters on responses. One simulation assumes that the element stiffnesses are random but that the loads are deterministic, equal to their expected values. The other simulation assumes that both stiffnesses and loads are random, with moments given by Eq. 27. Both simulations assume Gaussian random variables; 1,000 realizations of the random vector were generated to estimate response moments. Figures 6 and 7 show scatter plots, histograms, means, and covariances of responses of the random system to deterministic loads. The responses may be compared with those given by Eqs. 14 and 15, which were computed analytically for the deterministic system subject to random loads. The expected displacements are essentially equal, but the coefficients of variation and the correlation coefficient between the two displacements are much smaller. The displacement histograms are skewed toward larger absolute values of displacements.

Therefore displacement responses of linear elastic, random (Gaussian) systems to deterministic loads are not Gaussian, because displacements are functions of the inverses of the random element stiffnesses. Figure 7 shows that expected values of the element forces are essentially the same as those of the deterministic system subject to random loads, but the coefficients of variation are very small. The element forces are perfectly correlated if the loads are deterministic.

Figure 8
Nodal displacements - nondeterministic system and loads

\[
\begin{bmatrix}
0.90152 \\
0.00167
\end{bmatrix}
\]

\[
\nu_s = [0.6321 \\
0.3325]
\]

\[
\Sigma_u = \begin{bmatrix}
8.955 \times 10^6 & -47.377 \times 10^4 \\
-47.377 \times 10^4 & 297.74 \times 10^4
\end{bmatrix}
\]

\[
\rho_u = \begin{bmatrix}
1 & -0.9175 \\
-0.9175 & 1
\end{bmatrix}
\]
Figure 9

Element axial forces - nondeterministic system and loads

This result is specific to the case of deterministic loads on a node with three concurrent elements. The two nodal equilibrium equations constrain the magnitudes of the three element forces; given one element force, the other two must be deterministically related to it to satisfy equilibrium. If there were four or more elements converging to the node, the element forces would not be perfectly correlated. For a two-element, statically determinate truss, the element forces are determined by equilibrium. Therefore the element forces are deterministic and independent of the random element stiffnesses.

Figures 8 and 9 show corresponding responses for the case of random stiffnesses and loads. The expected displacements and forces are close to those of the deterministic system subject to random loads. The coefficients of variation are slightly larger and the correlation coefficients are slightly smaller. Although the system is nondeterministic, the displacement histograms are nearly symmetric because response variability arises largely from the random loads.

Observations. Any approximate analytical approach of simulation procedure for estimating responses of nondeterministic systems must satisfy the condition that element forces in statically determinate systems are independent of uncertainty in material and element properties. For hyperstatic systems, uncertainty in system parameters generally increases response coefficients of variation. Computed response moments of nondeterministic systems remain conditional on the defined system model and on the assumed load distribution.
Design based on nondeterministic linear elastic analyses

The text, *Structural and Civil Engineering Design* (Addis 1999) is an important compilation of articles on historical developments in structural design. The present view of the process is that a structural system is first conceptualized and then its acceptability is verified by iterative "detailed design." In current practice, detailed design involves defining load and system models and performing structural analyses to predict the effects of the loads. The effects are then compared with strength and serviceability design criteria. Strength is normally checked at the element level, using material and/or element strength models derived from testing and theory. These facets of detailed structural design developed gradually, starting with the theoretical advancements in linear elastic structural analysis by Navier (1826). Because of Navier's work, the dominant concept for achieving safety was to limit calculated stresses to fractions of measured material and element strengths. The work of Long (1836) and Mahan (1837) reflects Navier's influence in the U.S. The subsequent contributions of Rankine to such an "allowable stress design" (ASD) approach are discussed by Channell(1982) and Addis(1989–90). The limitations of ASD in turn motivated the development of "plastic design" in steel and "ultimate strength design" in concrete. These approaches achieve safety by using "load factors" and "resistance factors". Initially, single load factors were estimated by calibration with allowable stress design of statically determinate beams (Beedle 1957; Heyman 1957). However, it was understood that load factors also reflect uncertainty in the loads (Torroja 1958, ACI Committee 318 Report 1962). Therefore separate load factors for dead and live loads were introduced. For the recent development of the "load and resistance factor design" (LRFD) format in the U.S., it has been argued that the proposed factors correspond to target element reliabilities (Ravindra and Galambos 1978; Ellingwood et al. 1982). However, an LRFD format cannot assure uniform target element reliabilities because, with random loads, the variances of the load effects generally differ, as demonstrated by the nondeterministic analysis of the basic Navier truss. Although LRFD uses formulas that model element strength, factored load effects are usually computed using linear elastic analyses.

In 1967–69, Cornell suggested a probabilistic design format based on expected values and coefficients of variation of load effects and resistances (Cornell 1969). He did not advocate use of second-moment, nondeterministic linear elastic analyses to compute coefficients of variation of load effects. Rather, he suggested using coefficients of variation of load effects estimated from those of the "total load" and from "uncertainty in the structural analysis". Cornell's approach of explicitly estimating coefficients of variation of both resistances and
load effects is in fact what is needed to achieve uniform target element reliabilities. Considering the element strength design criterion, uniform target element reliabilities may be achieved by using a second moment reliability index, \( \beta \) (Cornell, 1969). For an element resistance, \( R \), and a corresponding uncorrelated load effect, \( S \), the reliability index is given by:

\[
\beta = \frac{E[R] - E[S]}{\sqrt{\text{var}[R]} + \text{var}[S]}
\]

(28a)

Or, introducing coefficients of variation, \( V_R \) and \( V_S \),

\[
\beta = \frac{E[R] - E[S]}{\sqrt{(V_R^2E[R])^2 + (V_S^2E[S])^2}}
\]

(28b)

Eq. 28b is directly suitable for design. That is, \( E[S] \) and \( V_S \) may be computed using a second-moment nondeterministic linear elastic analysis, although such calculated \( V_S \) must be increased to capture all the uncertainties discussed by Toroja (1958) and Cornell (1969). \( V_R \) is estimated from statistical data on material strength and from uncertainties due to “fabrication” and “professional assumptions” (Cornell 1969). A target element reliability index, \( \beta \), is prescribed. Then the required \( E[R] \) may be computed from (quadratic) Eq. 28b. Rather than using Eq. 28b directly, an equivalent alternate procedure may be used as follows. A “reduced” reliability index, \( \beta_s \), is defined by introducing the “triangle inequality” approximation for the denominator in Eq. 28b:

\[
\beta_s = \frac{E[R] - E[S]}{V_R E[R] + V_S E[S]}
\]

(29)

\( \beta_s \leq \beta \) because the denominator in Eq. 29 is an upper bound to the actual square root in Eq. 28b. Rearranging Eq. 29 and introducing the design inequality, yields:

\[
E[R](1 - \beta_s V_R) \leq E[S](1 + \beta_s V_S)
\]

(30)

The above equation, which was first derived by Lind (1971) using a different method, may be used to determine \( E[R] \). The value of \( \beta_s \) that corresponds to a target element reliability, \( \beta \), may be determined by substituting the expression for \( E[R] \) from Eq. 30 into Eq. 28b. The substitution leads to the following quadratic equation for \( \beta_s \) in terms of \( \beta \) and the coefficients of variation:

\[
(2V_S^2V_R^2 - (V_S + V_R)^2/\beta^2)\beta_s^2 + 2V_S V_R (V_R - V_S) \beta_s + V_S^2 + V_R^2 = 0
\]

(31)
Therefore an equivalent alternate design procedure involves solving for $\beta_s$ from the above equation and then determining $E[R]$ from the simpler Eq. 30.

For example, the required expected values of the strengths of the three elements in the Navier truss may be determined as follows. From the simulation of the nondeterministic system, the coefficients of variation of the axial force responses are:

$$V_{N1} = 0.25885$$
$$V_{N2} = 0.29526$$
$$V_{N3} = 0.34304$$

Assume $V_R = 0.1$ and prescribe a target element reliability index $\beta = 4$. Then, from Eq. 31, the three $\beta_s$ values are:

$$\beta_{N1} = 2.830$$
$$\beta_{N2} = 2.835$$
$$\beta_{N3} = 2.845$$

Using Eq. 30, the required expected values of strength are:

$$E[R_{N1}] = 2.416E[N_{N1}]$$
$$E[R_{N2}] = 2.564E[N_{N2}]$$
$$E[R_{N3}] = 2.762E[N_{N3}]$$

These expected element strength values provide uniform target element reliabilities for the computed coefficients of variation of responses. However, the system reliability is not explicitly quantified.

A similar approach may also be used for serviceability limit states such as, say, for displacements. "Limit displacements" should be prescribed in terms of expected values and coefficients of variation. Then, given the computed moments of displacement responses, second moment system (and element) serviceability reliability indices may be computed. If the serviceability reliability indices are smaller than the target values, the element stiffnesses must be increased.

**Mechanism limit state analyses. Deterministic systems and loads**

The design of a structure should consider its safety at all stages of its life: during construction, in normal usage, under extreme design events, under repeated cyclic actions, and in various stages of environmental degradation. An engineer must imagine potential failure modes and design to achieve an acceptable reliability in all potential failure scenarios. One potential limit state of a structure
occurs by static overloading, when a set of elements yield or rupture or a set of "plastic hinges" form such that a part or all of the structure becomes a mechanism, allowing large kinematic motions without increases in load. Controlling the likelihood of such a failure mode can form a basis for structural design, in addition to the criterion of assuring satisfactory performance during normal conditions.

How a structure reaches a failure mechanism limit state depends on the actual behavior of its components or members, which in turn depends on their stability and on material behavior. There are many linear or nonlinear, time-dependent or time-independent, one- or multi-dimensional material constitutive models. Time-independent, one-dimensional material models are sufficient for defining the static behavior of axial (truss) elements and plane frame beam elements. If the elements are stable, then their strength behavior is the same as that of the material. Figure 10 shows basic one-dimensional models for (stable) axial elements, which correspond to their one-dimensional stress-strain material models. In addition to the basic elastic-plastic and elastic-brittle models, Figures 10c and 10d show models for prestressed, tension-only (cable) or compression-only (contact) duc-

![Force-deformation models](image-url)
tile elements. The behavior of a structure is fundamentally different depending on whether the elements are ductile or brittle. If a brittle element breaks, its shed force must be redistributed to the remaining structure. If a ductile element yields, its stiffness becomes zero but it maintains its force through additional deformation. Analyses for predicting strength in these two conditions differ.

**Ductile elements.** Heyman (1998) provides a historical perspective on the development of procedures for predicting the strength of framed structures consisting of (stable) ductile elements. Such limit state analyses are usually conditional on a proportional loading. The objective is to determine the load factor corresponding to the unique collapse mechanism that satisfies equilibrium and the yield condition. Historically, two manual approaches, the mechanism method and the equilibrium method, evolved on the basis of the upper and lower bound theorems. Current structural analysis programs increment either a load or a displacement and trace the sequential development of element yielding or plastic hinge formation until the stiffness matrix becomes singular. Herein, for purposes of comparing deterministic and nondeterministic limit state analyses, the manual "mechanism method" is used. That is, all the possible failure mechanisms of the Navier truss are enumerated and the lowest load factor is associated with the unique controlling mechanism. The system consisting of elements with both tensile and compressive strengths is considered first. The set of equilibrium equations interpretable as limit state functions may be determined by prescribing compatible, virtual deformations and displacements that define a mechanism and then invoking the Principle of Virtual Displacements. The set of limit state functions for the Navier truss is as follows:

**Elements 1 and 2 yield:**

\[ g_1 = \frac{14}{13} R_n + \frac{7}{2\sqrt{2}} R_{t2} - \left( \frac{3}{4} Q_1 + Q_2 \right) \]  \hspace{1cm} (32a)

**Elements 1 and 3 yield:**

If \(-Q_1 + Q_2\) is positive

\[ g_2 = \frac{7}{13} R_{c1} + \frac{7}{3} R_{t3} - (-Q_1 + Q_2) \]  \hspace{1cm} (32b)

If \(-Q_1 + Q_2\) is negative

\[ g_3 = \frac{7}{13} R_n + \frac{7}{5} R_{c3} + (-Q_1 + Q_2) \]  \hspace{1cm} (32c)
Elements 2 and 3 yield:

If \( \frac{5}{12} Q_1 + Q_2 \) is positive

\[
g_4 = \frac{7}{12\sqrt{2}} R_{n1} + \frac{14}{15} R_{n2} - \left( -\frac{5}{12} Q_1 + Q_2 \right)
\]  \hspace{1cm} (32d)

If \( \frac{5}{12} Q_1 + Q_2 \) is negative

\[
g_5 = \frac{7}{12\sqrt{2}} R_{c1} + \frac{14}{15} R_{c2} - \left( -\frac{5}{12} Q_1 + Q_2 \right)
\]  \hspace{1cm} (32e)

For any prescribed proportion between \( Q_1 \) and \( Q_2 \), there are only three possible mechanisms. For example, consider the following deterministic values of load and strengths:

\[
\begin{bmatrix}
Q_1 \\
Q_2
\end{bmatrix} = \begin{bmatrix} 0.2 \\ 1.0 \end{bmatrix} Q, \quad \begin{bmatrix}
R_{n1} \\
R_{n2}
\end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.2 \end{bmatrix} R_p, \quad \begin{bmatrix}
R_{c1} \\
R_{c2}
\end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix} R_q
\]  \hspace{1cm} (33)

Then the three possible limit state functions are \( g_1 \), \( g_2 \), and \( g_4 \). The controlling mechanism corresponds to limit state function \( g_1 \); that is, elements 2 and 3 yield in tension at a critical load factor \( Q/R_q = 1.965 \). If \( R_q = 80 \text{kN} \), then a mechanism will form when \( Q = 1.965(80) = 157 \text{kN} \).

Now consider the Navier truss again, but as shown in figure 11, with two cable and one contact element, prestressed with initial forces \( N_{1o} \), \( N_{2o} \), and \( N_{3p} \). Because they are tension-only and compression-only elements, \( R_{e1} = 0 \), \( R_{e2} = 0 \), and \( R_{c2} = 0 \).

Substituting these zero strengths in Eqs. 32 gives the following modified limit state equations:

\[
g_1 = \frac{14}{13} R_{n1} - \left( \frac{3}{4} Q_1 + Q_2 \right)
\]  \hspace{1cm} (34a)

\[
g_2 = \frac{7}{3} R_{n2} - \left( -Q_1 + Q_2 \right)
\]  \hspace{1cm} (34b)
Figure 11
Navier truss with tension-only and compression-only elements

\[ g_3 = \frac{7}{13} R_n - \left( -\frac{5}{12} Q_1 + Q_2 \right) \]  \hspace{1cm} (34c)

\[ g_4 = \frac{14}{15} R_n - \left( -\frac{5}{12} Q_1 + Q_2 \right) \]  \hspace{1cm} (34d)

\[ g_5 = \frac{7}{12\sqrt{2}} R_n - \left( -\frac{5}{12} Q_1 + Q_2 \right) \]  \hspace{1cm} (34e)

Because tension-only and compression-only elements are used, only one element contributes to the internal virtual work when a mechanism is formed. With the assumed deterministic strengths and loads given by Eq. 33, the three appropriate limit state functions are \( g_1, g_2, \) and \( g_5. \) The controlling mechanism corresponds to \( g_3, \) which occurs at a load factor \( Q/R_n = 0.936, \) less than half of the load factor for the truss with elements having both tensile and compressive strengths.

**Observations.** In addition to the central concepts contained in the limit theorems and in the uniqueness theorem, deterministic limit state analyses provide other insights on behavior. For one, relative element stiffnesses and elastic distributions of forces do not affect strength. Nor is strength affected by an initial state of self-equilibrated forces. Therefore initial stresses from lack-of-fit or from small support movements do not affect strength. A system of ductile elements may be viewed as a set of “ductile failure modes in series”. The strength of the system is controlled by the “weakest failure mode”. Deterministic limit state analyses do not provide checks on performance at service loads. The analyses are conditional on a load proportion and on a set of deterministic strengths. They do not provide quantitative estimates of the probability of failure in a mechanism limit state mode.
**Brittle elements.** Consider now the Navier truss with brittle elements as depicted in figure 10b. Four possible load displacement behaviors of the system under load and displacement control are shown in figure 12. In general, a hyperstatic brittle system may attain its maximum load or strength at the failure of the first, second, or nth element. After failure of one or more elements, the system remains linear elastic, but with reduced stiffness. Consider the Navier
truss with the deterministic loads and strengths prescribed in Eq. 33. The forces in the elements in all possible states before a mechanism is formed are as follows:

All three elements active

\[
\begin{bmatrix}
N_1 \\
N_2 \\
N_3
\end{bmatrix} = \begin{bmatrix}
0.3091 \\
0.6603 \\
0.6903
\end{bmatrix} Q
\] (35a)

Element 1 broken

\[
\begin{bmatrix}
N_1 \\
N_2
\end{bmatrix} = \begin{bmatrix}
0.9293 \\
0.5714
\end{bmatrix} Q
\] (35b)

Element 2 broken

\[
\begin{bmatrix}
N_1 \\
N_2
\end{bmatrix} = \begin{bmatrix}
1.0679 \\
0.9821
\end{bmatrix} Q
\] (35c)

Element 3 broken

\[
\begin{bmatrix}
N_1 \\
N_2
\end{bmatrix} = \begin{bmatrix}
-1.4857 \\
2.2223
\end{bmatrix} Q
\] (35d)

Setting \([N_1, N_2, N_3]^T\) equal to the element strengths, it is determined that as the load is increased element 2 breaks first at a load factor \(Q/R_n = 1.817\). Considering the remaining \([N_1, N_3]^T\) forces under load control, it may be inferred that a second element breaks immediately and that a mechanism is formed. The strength of the brittle system is therefore \(Q/R_n = 1.817\), a value that is smaller than that of the ductile system.

Consider next the brittle system with a set of initial, self-equilibrated forces as follows:

\[
\begin{bmatrix}
N_{19} \\
N_{20} \\
N_{30}
\end{bmatrix} = \begin{bmatrix}
0.5000 \\
-0.4350 \\
0.1935
\end{bmatrix} R_n
\] (36)
superposing forces in the initial condition with all elements active gives:

\[
\begin{bmatrix}
N_1 \\ N_2 \\ N_J
\end{bmatrix} =
\begin{bmatrix}
0.3091 \\ 0.6603 \\ 0.6903
\end{bmatrix}Q +
\begin{bmatrix}
0.5000 \\ -0.4350 \\ 0.1935
\end{bmatrix}R_u
\]

(37)

In this case, element 1 will break first at a load factor \( Q/R_u = 1.6176 \). Then considering equilibrium under load control with element 1 broken, element 2 will break immediately thereafter. Therefore the strength of the brittle system with the prescribed set of initial self-equilibrated forces is \( Q/R_u = 1.6176 \), lower than the strength of both the ductile system and of the brittle system with zero initial forces. Therefore an initial self-equilibrated state of stress can decrease the strength of a hyperstatic brittle system.

**Mechanism limit state analyses. Nondeterministic strengths and loads**

There are several conceptual differences between deterministic and nondeterministic limit state analyses of framed structures. If strengths and loads are considered to be random variables, there is no "proportional loading", no unique controlling mechanism, and no minimum load factor that defines the system strength. Rather, model and system "reliability indices" or probabilities of failure are computed. As for the case of deterministic limit state analyses, nondeterministic limit state analyses differ depending on whether the elements are ductile or brittle.

**Ductile elements with both tensile and compressive strengths.** The principal concepts of nondeterministic limit state analyses of ductile systems may be illustrated by again considering Navier's truss. In the context of nondeterministic analysis, the strength and load quantities in the limit state equations (Eqs. 32) are viewed as a vector of basic random variables, \( \mathbf{X} = [R_{T1}, R_{T2}, R_{T3}, R_{C1}, R_{C2}, R_{C3}, Q_1, Q_2] \). A random vector may be completely defined by a multidimensional probability density function. Alternatively, the vector may be partially described by its mean and covariance matrices, \( \mathbf{X} \sim (\mu_X; \Sigma_X) \). Such a second moment partial description of the basic random variables allows probabilistic limit state analyses as outlined in figure 13. First, the vector \( \mathbf{X} \) is transformed into standard space (variables with zero means, unit variances, and zero covariances) using the algorithm given by Rubinstein (1981). The hyperplane modal limit state equations, \( g(\mathbf{X}) \), are also transformed into standard space and denoted as \( g^*(\mathbf{U}) \). A second-moment modal reliability index, \( \beta_1 \), is defined for each mode \( g_i(\mathbf{U}) \) (Cornell 1969; Veneziano 1974) Figure 13 shows the meaning of \( \beta_1 \) in two-dimensional space; that is, \( \beta_1 \) is the (minimum) distance from the origin in standard space to the limit state hyper-
plane, \( g_i(U) \). The point on \( g_i(U) \) closest to the origin, denoted by \( U_i^* \), is called a "design point". \( \beta_i \) is a scalar second moment index of modal reliability. Recognizing that the structure may be viewed as a set of ductile modes in series, the system is safe only if none of the failure modes occur. Hasefer and Lind (1974) proposed the minimum of the modal \( \beta_i \)'s, denoted as \( \beta_{\text{HL}} \), as a scalar second moment index of system reliability. For hyperplane modal limit state functions, the modal reliability indices, \( \beta_i \), may be computed analytically. As an example, assume \( X \) to have the following moments:

\[
\begin{bmatrix}
R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\
R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\
R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
R_{15} & R_{25} & R_{35} & R_{45} & R_{55}
\end{bmatrix}
\begin{bmatrix}
80 \\
40 \\
96 \\
48 \\
112 \\
56 \\
10 \\
50
\end{bmatrix}
\begin{bmatrix}
64 & 25.6 & 38.4 & 19.2 & 44.8 & 22.4 & 0 & 0 \\
25.6 & 16 & 19.2 & 9.6 & 22.4 & 11.2 & 0 & 0 \\
38.4 & 19.2 & 92.16 & 36.86 & 53.76 & 26.88 & 0 & 0 \\
19.2 & 9.6 & 36.86 & 23.04 & 26.88 & 13.44 & 0 & 0 \\
44.8 & 22.4 & 53.76 & 26.88 & 125.44 & 50.176 & 0 & 0 \\
22.4 & 11.2 & 26.88 & 13.44 & 50.176 & 31.36 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 15 \\
0 & 0 & 0 & 0 & 0 & 0 & 15 & 225
\end{bmatrix}
\]

(38)

The form of \( \Sigma_{xx} \) implies that the loads are assumed to be uncorrelated with the strengths. The variances of the strengths were computed assuming C.O.V. \( \rho_r = 0.1 \). The covariances between any two strengths were computed assuming \( \rho_{\text{RSS}} = 0.5, \rho_{\text{RSR}} = 0.5, \rho_{\text{RRS}} = 0.5, \) and \( \rho_{\text{RSS}} = 0.8 \). For the loads, it was assumed that C.O.V. \( \rho_{\text{L}} = 0.2 \), C.O.V. \( \rho_{\text{L}} = 0.3 \), and \( \rho_{\text{CSS}} = 0.5 \). Following the steps shown in Figure 13, the second moment modal reliability indices were computed to be: \( \beta^* = \{6.19, 6.29, 9.13, 5.05, 7.38\} \). The Hasefer-Lind system reliability index is: \( \beta_{\text{HL}} = \beta_{\text{max}} = \beta_A = 5.05 \). The "most likely" mode is 4, which is also the controlling mode from the deterministic limit state analysis. The meaning of \( \beta_{\text{HL}} = 5.05 \) in terms of system reliability may be determined by estimating the system reliability numerically, say by simulation. Crude or "brute force" Monte Carlo simulation involves assuming a probability density function for the variables, generating samples of the random vector, \( U \), and computing the fraction of the total number of realizations that occur "outside" the convex limit state surface of the system. When probabilities of failure are small, it is necessary to use "variance reduction" statistical techniques such as "importance sampling" (Ross 1990) in order to decrease the number of realizations required for a good estimate of system reliability. Assuming normal random variates and using Monte Carlo simulation with importance sampling, it is estimated that \( \text{P}[\text{failure}] = 2.26 \times 10^{-7} \). Denoting \( \Phi(u) \) as the cumulative distribution function for a standard normal variate, it can be noted that \( 1 - \Phi(5.05) \), is approximately \( 2 \times 10^{-7} \), which
- Second moment vector of basic random variables
  \[ X \sim (\mu_X; \Sigma_{xx}) \]
  \[ U \sim (0, I) \]
- Hyperplane modal limit state static equations that define a convex failure surface
  \[ g(X) = AX \]
  \[ g(U) = BU + C \]
- Scalar hyperplane modal limit state equation
  \[ g_i(X) = a_{1i}x_1 + a_{2i}x_2 + \ldots + a_{ni}x_n \]
  \[ g_i(U) = b_{1i}u_1 + b_{2i}u_2 + \ldots + b_{ni}u_n + c_i \]
  \[ g_i(U) = b_i^T U + c_i \]
- Scalar, second moment, modal reliability index, \( \beta_i = \frac{c_i}{\sqrt{\sum_{j=1}^{n} b_j^2}} \)

![Diagram](image)

Figure 13
Second moment reliability analysis

is close to the estimated probability of system failure. This indicates that mode 4
"contributes the most" to the probability of system failure and that

\[ \beta_{\text{HL}} = \beta_{\text{MN}} = \beta_4 = 5.05 \] is a good indicator of system reliability. If several
modes had \( \beta_i \) values slightly larger than 5.05, then the probability of system failure
would increase whereas \( \beta_{\text{HL}} \) would not. Basically, \( \beta_{\text{HL}} \) relates system reliability to
the reliability of the weakest mode.

Many studies may be performed by changing the moments of \( X \) and noting
changes in the second moment modal reliability indices and in the probabilities
of system failure. For example, consider the following new moments of \( X \):
In which the expected values of the resistances have been reduced, it is now found that $\beta^T = [4.092 \ 4.187 \ 7.682 \ 2.652 \ 5.998]$. Therefore, the most likely mode remains mode 4 but now $\beta_{RL} = \beta_{eq} = \beta_{s} = 2.652$. Assuming normal random variables, the probability of failure estimated by Monte Carlo simulation is: $P[\text{failure}] = 0.004$. This value is again close to $1 - \Phi(2.652)$, the probability of failure of mode 4. Assuming lognormal random variables having the same mean and covariance matrices, it is estimated by Monte Carlo simulation that $P[\text{failure}] = 0.0127$. Therefore, the Hasofer-Lind second moment system reliability index, $\beta_{RL}$, should not be associated with a unique probability of system failure.

It is of interest to study the effects of correlation between element resistances on system reliability. Assuming $\rho = 0.999$ between all pairs of (normal) resistances it is estimated by Monte Carlo simulation that $P[\text{failure}] = 0.005$. Assuming $\rho = 0.0$ between all pairs of resistances, it is estimated that $P[\text{failure}] = 0.0031$. Therefore, system reliability decreases as the ductile resistances become more correlated. Although the system may be viewed as a series combination of modes, each mode requires parallel yielding of elements. It is the reliability of a mode consisting of ductile elements in parallel that decreases as the correlation between resistances increases. And this decrease in modal reliabilities decreases the system reliability.

**Ductile elements with only tensile or compressive strengths.** Consider now the nondeterministic limit state analysis of the Navier truss with tension-only and compression-only elements as shown in figure 11. That is, the elements have deterministic zero strengths: $R_{x1} = R_{y1} = R_{x2} = 0.0$. For such a structure, there are five basic random variables: $X^T = [R_{x1} \ R_{y1} \ R_{x2} \ Q_1 \ Q_2]$. The moments given by Eq. 38 reduce to:

$$X \sim \left( \begin{array}{cccccc}
60 & 36 & 14.41 & 18 & 9 & 21 \\
30 & 14.41 & 9 & 9 & 4.5 & 10.5 \\
60 & 18 & 9 & 36 & 14.4 & 21 \\
30 & 9 & 4.5 & 14.4 & 9 & 10.5 \\
70 & 21 & 10.5 & 21 & 10.5 & 5.25 \\
35 & 10.5 & 5.25 & 10.5 & 5.25 & 19.6 \\
10 & 4 & 0 & 0 & 0 & 4 \\
50 & 0 & 0 & 0 & 0 & 15
\end{array} \right)$$

(39)
A second moment reliability analysis gives: \( \beta^* = [1.59, 5.54, 5.63, 3.27, 4.45] \).

Therefore \( \beta_{ll} = \beta_{\text{min}} = \beta_1 = 1.59 \). Mode 1, which was the controlling mode in the corresponding deterministic limit state analysis, is now the most likely mode.

The second moment system reliability index has decreased from 5.05 to 1.59 with the use of tension-only and compression-only elements. Using the smaller mean resistances given in Eq. 39, modified for the zero strengths, yields the following modal reliability indices:

\[ \beta^* = [0.417, 3.38, 4.998, 1.22, 3.97] \]. Assuming normal random variables, it is estimated by Monte Carlo simulation that \( P[\text{failure}] = 0.340 \). Therefore the use of tension-only and compression-only elements increased the estimated probability of failure from 0.004 to 0.34.

**Brittle elements.** In general, simulation is required to estimate the reliability of parallel-brittle systems. For each realization of the basic random variables, an incremental structural analysis with element deletion and force redistribution must be performed to determine if system failure occurs. The system reliability is estimated from the fraction of realizations that do not lead to failure. For the basic Navier truss, the set of (four) mutually exclusive survival modes may be enumerated: it can survive with all elements intact, or with only elements 1 and 2 intact, or with only elements 1 and 3 intact, or with only elements 2 and 3 intact.

The probability of survival of any one mode may be expressed in terms of probability inequalities that follow from structural analysis of the system in the damaged condition associated with the particular survival mode. For example, the probability of survival with only elements 2 and 3 intact may be expressed as:

\[
P[\text{system survival with only elements 2 and 3 intact}] = P[\text{Failure element 1 in tension} \cap \text{survival of element 2 in tension} \cap \text{survival of element 3 in both tension and compression}]
\]

\[
P \left[ R_{R_1} - (0.421Q_1 + 0.225Q_2) \leq 0 \cap R_{R_2} - \left( \frac{3\sqrt{3}}{7} Q_1 + \frac{4\sqrt{2}}{7} Q_2 \right) > 0 \cap R_{R_3} - \left( -\frac{5}{7} Q_1 + \frac{5}{7} Q_2 \right) > 0 \cap R_{R_4} - \left( -\frac{5}{7} Q_1 + \frac{5}{7} Q_2 \right) > 0 \right]
\]  \hspace{1cm} (41)

The probability of system survival in the other modes may be expressed in a similar way.

Because the modes are mutually exclusive, the probability of system survival is the sum of the probabilities of survival in each mode. In a Monte Carlo simulation, for each realization of the vector of basic variables, the inequalities associated with all four modes are checked to determine if the system survives. The fraction:...
of realizations in which the system fails provides an estimate of the probability of failure. Simulations were performed for the parallel-brittle Navier truss with normal basic random variables with means and variances given by Eq. 39 and different assumptions on the correlations between element resistances. Table 1 compares the estimated probabilities of failure of the parallel-brittle system with those of the parallel-ductile system. The reliabilities of the parallel-ductile and parallel-brittle systems approach one another as the correlations between element resistances increase, but the parallel-ductile system remains more reliable.

<table>
<thead>
<tr>
<th>Correlation between resistances</th>
<th>Estimated probability of failure of a parallel-brittle system</th>
<th>Estimated probability of failure of a parallel-ductile system</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0107</td>
<td>0.0031</td>
</tr>
<tr>
<td>0.5, 0.8</td>
<td>0.0097</td>
<td>0.004</td>
</tr>
<tr>
<td>0.999</td>
<td>0.008</td>
<td>0.005</td>
</tr>
</tbody>
</table>

Table 1
Estimated probabilities of failure of parallel-brittle and parallel-ductile systems

Strength design of ductile systems based on mechanism limit state analyses

With deterministic mechanism limit state analyses, the principal design objective is to provide element resistances such that collapse loads are greater than factored design loads. For example, the Navier truss collapse load, \( Q = 1.965P_{cr} \), should be greater than the factored design load. Other than increasing individual element resistances to obtain collapse loads greater than factored loads, there is no one, commonly-used, "detailed design" method that "optimizes" a design based on deterministic mechanism limit state analyses. A general objective of detailed design is to preclude "local" mechanisms that involve only one or a few elements. Heyman (1951) and Zeman and Irvine (1986) have defined algorithms to achieve minimum-weight optimal designs based on deterministic mechanism limit state analyses.

With nondeterministic mechanism limit state analyses, the principal design objective is to provide (expected values of) resistances such that the system reliability is acceptable. For small structures such as Navier's three-bar truss, it is possible to enumerate all failure modes, write explicit limit state functions, and determine the most likely failure mode. However, for most realistic structures, it is not possible to enumerate all the potential failure modes and write explicit limit state functions. In general, extensive simulations involving multiple incremental
limit-state analyses are needed to estimate reliability. A general detailed design process for optimally modifying individual elements to affect system reliability remains to be defined. Research on such “reliability-based optimal design” methods is ongoing (Frangopol 1997).

Summary and observations

The development of analysis-based design in the early 19th century revealed the uncertainties in modeling systems and live loads as well as the need for a method to achieve reliability, for both strength and serviceability. Discussion of these issues was largely quieted by the successful adoption of deterministic linear elastic analyses together with very conservative “working” live loads and allowable fractions of conservative material/element strengths. Deterministic linear elastic analyses have provided invaluable insights on structural behavior and have served as bases for the safe development of innovative structural forms. A very useful property of linearity is that any load effect vector may be expressed as a linear combination of the magnitudes of actions such as applied loads and prescribed support displacements. Although deterministic linear elastic analyses provide only “point estimates” of responses, the associated allowable stress design methods have produced safe designs.

The limitations of linear elastic analysis methods motivated the development of mechanism limit state analyses. The limit theorems, the uniqueness theorem, and the fact that an initial self-equilibrated state of stress does not affect the strength of systems of ductile elements are invaluable insights on structural behavior. With the introduction of “load factors”, design methods based on deterministic limit state analyses have also provided safe designs for strength. Such design methods, however, must generally include separate checks on serviceability limit states.

Deterministic linear elastic and mechanism limit state analyses do not explicitly consider uncertainty in loads or in the material and element properties used in system models. Therefore element and system reliabilities are unknown, except that the historic reliabilities of systems designed by deterministic methods have, in general, been accepted. However, a better understanding of the factors that control reliability can improve designs. And it is rational to develop design methods that can provide uniform element reliabilities and quantifiable system reliability. It is these objectives that have motivated research on non-deterministic structural analysis and design methods.

Nondeterministic linear elastic analyses have been formulated by defining quantities as random variables, interval numbers, and fuzzy numbers. The most advanced development is associated with random-variable-based models; more specifically, with “second moment” models that use mean and covariance matri-
ces to define random variables. The complexity of nondeterministic linear elastic models depends on whether only the loads, or only the system, or both the loads and the system are modeled using random variables. If the system is considered deterministic, then the computation of the mean and covariance matrices of any response vector is straightforward. If both the system and loads are modeled using random variables, then some analytical approximations and/or Monte Carlo simulation is required to estimate mean and covariance matrices of responses. In both cases, the "radical" departure is that the loads are defined by mean and covariance matrices rather than "working" values.

Mean and covariance matrices of responses provide useful design information. A variance is a measure of the uncertainty in a response and a covariance is a measure of the linear correlation between two responses. The correlation between two responses is relevant to the design of an element for two or more load effects such as axial force and bending moment. The correlation indicates the likelihood that a pair of load effects may both have high values. Cornell (1969) proposed a design method based on estimating means and coefficients of variation of load effects and element resistances. The method provides for the design of elements with uniform target second moment reliability indices for strength. The method may also be used to attain a prescribed system reliability index for serviceability.

Nondeterministic mechanism limit state analyses do not use the concept of "proportional loading". There is no unique controlling mechanism and minimum load factor. Rather, a structure is viewed as a series system of failure modes. Depending on element behavior, a failure mode can in turn be considered as a parallel-ductile or parallel-brittle subsystem. Nondeterministic mechanism limit state analyses estimate reliability indices or probabilities of failure for the modes of a system. Nondeterministic limit state analyses quantify the effects of element behavior (ductile or brittle), element strength distribution, and correlations between random variables on modal and system reliability. In general, it is not possible to enumerate all failure modes. Thus failure surfaces have to be estimated and extensive simulation is required.

In summary, nondeterministic models may be said to be "more rational" in the sense that they model some of the real uncertainties in loads and systems. There are practical, well-developed, random-variable-based analysis/design methods, at least for linear elastic systems. These analysis/design methods provide useful insights on structural behavior. However, the impact of such procedures on the reliability and economy of designs must still be determined. In addition, for implementation, considerable change must occur in current analysis/design practice. Mean and covariance matrices of actions, system properties, and element resistances must be defined. Nondeterministic analysis
algorithms must be implemented in commonly-used finite element programs. Nondeterministic design procedures must be codified and accepted by legal entities. Most importantly, practicing design engineers must understand nondeterministic methods and be convinced of the merits of applying them.

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Reference list


